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THESIS

6 EFFICIENT ESTIMATION OF NEGATIVE BINOMIAL PARAMETERS USING EMPIRICAL LA PLACE TRANSFORM

by 9 Master's thesis,

10 Resai Caglayan

11 Sep~~tember~~ 1978 12 68p.

Thesis Advisor: R.R. Read

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Efficient Estimation of Negative Binomial Parameters Using Empirical La Place Transform

by

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Submitted in partial fulfillment of the requirements for the degree of

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ABSTRACT

A new method, based on empirical La Place transform, was developed to find asymptotically efficient estimates of negative binomial distribution parameters. These estimates were found fairly close to those found by the method of maximum likelihood. Efficiencies over 95 percent were obtained. The method was tested with a set of data, generated by computer, and found to be satisfactory except in a few cases. Maximum likelihood also fails to be satisfactory in these cases.

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I. INTRODUCTION

The need for readily computed parameter estimates is great. Maximum likelihood estimators are known to be asymptotically efficient, but in many settings they are hard to find. The most popular alternative is the method of moments which usually yields readily computed estimates, but sometimes these estimates are not very efficient. See ref. [2]. This study looks at the efficiency of a method which uses the probability generating function (empirical La Place transform evaluated at a desirable value of its argument, u_o).

The method presented herein requires computing power greater than that of the method of moments, but less than that of maximum likelihood, which calls for a psi function capability. It can be used with a hand held calculator that has the lower order transcendental functions, i.e., logarithm, roots and power.

The basic idea is to select a system of estimating equations which equates various statistics to their expected values. In chapter II, estimating equations were set and efficiency computations were done using theoretical work presented in [3]. Chapter III contains the procedure to be followed, applications of the method to some data sets, and comparisons with the other methods.

In chapter IV, efficiency and optimum u_o and t_o ($=-\log u_o$) tables for various p-r combinations are given and efficiency contours in three different planes are graphed.

II. METHODOLOGY

A negative binomial random variable X has the probability function

$$f(X=x;r,p) = \frac{\Gamma(r+x)}{x!\Gamma(r)} q^x p^r \qquad (2.1)$$

for $x = 0,1,2, \ldots \quad 0 < r,\ 0 < p < 1,\ p + q = 1$. The partial derivatives of its logarithm are, using $h = \log f$

$$\frac{\partial h}{\partial p} = \frac{r}{p} - \frac{x}{q}$$

$$\frac{\partial h}{\partial r} = \Psi(r+x) - \Psi(r) + \log p$$

Using the basic recursive formula for the psi function [1]

$$\Psi(r+x) - \Psi(r) = \sum_{j=1}^{x} \frac{1}{r-1+j}$$

one may then express the system of maximum likelihood equations as

$$\bar{x} - \frac{rq}{p} = 0$$

$$(2.3)$$

$$\underset{i=1,\ldots,n}{\text{Ave}} \sum \frac{1}{r-1+j} + \log p = 0$$

where the values $x_1, \ldots, x_n$ are the data that result when the population (2.1) is sampled n times. The system (2.3) is nonlinear in r,p and difficult to solve. Iterative methods must be used and the second member of (2.3) (or some version thereof) must be recomputed in each cycle. This is the main computational difficulty in using maximum likelihood in this setting.

The information matrix Λ, which is defined as:

$$\Lambda = -E \begin{bmatrix} \dfrac{\partial^2 h}{\partial p^2} & \dfrac{\partial^2 h}{\partial p \partial r} \\ \\ \dfrac{\partial^2 h}{\partial r \partial p} & \dfrac{\partial^2 h}{\partial r^2} \end{bmatrix}$$

is needed for efficiency calculations and can be developed using the methods presented in [2]. The result is

$$\Lambda = \begin{bmatrix} \dfrac{r}{qp^2} & -\dfrac{1}{p} \\ \\ -\dfrac{1}{p} & \Lambda_{22} \end{bmatrix}$$

where

$$\Lambda_{22} = \Psi'(r) - E(\Psi'(r+x))$$

The determinant of Λ is found (using [2]) as

$$|\Lambda| = \frac{1}{q^2} \sum_{n=1}^{\infty} \frac{q^n}{(n+1)} \cdot \frac{r!n!}{(r+n)!} \tag{2.4}$$

The two estimating equations g_1 and g_2, which are exploited herein, are given as

$$g_1(r,p;\vec{x}) = \bar{x} - \frac{rq}{p} = 0$$

$$(2.5)$$

$$g_2(r,p;\vec{x}) = \overline{t^x} - \left(\frac{p}{1-qt}\right)^r = 0$$

where

$$\overline{t^x} = \frac{1}{n} \sum_{i=1}^{n} t^{x_i}$$

is the empirical generating function. (Note: the substitution $t = \exp(-u)$ converts this to La Place transform). Reduction to a single equation is effected by solving $g_1 = 0$ for p in terms of $\bar{x}$ and r,

$$p = \frac{r}{\bar{x} + r} \tag{2.6}$$

and substituting this into g_2. A function $f(r)$ is obtained in terms of known variables, namely $\overline{t^x}$, $\bar{x}$, and t,

$$f(r) = \left(\frac{r}{r + \bar{x}(1-t)}\right)^r - t^{\bar{x}} = 0 \qquad (2.7)$$

We can solve equation (2.7) for r by Newton's method, the iterative relationship being

$$r_{i+1} = r_i - \frac{f(r_i)}{f'(r_i)}$$

where

$$f'(r) = \left(\frac{r}{r + \bar{x}(1-t)}\right)^r \cdot \left[\left(\frac{\bar{x}(1-t)}{r + \bar{x}(1-t)}\right) + \log\left(\frac{r}{r + \bar{x}(1-t)}\right)\right]$$

assuming an initial value of r and a suitable t can be found (see table 4.2). After solving for r, p can be found using (2.6).

Efficiency for this r-p pair is given by (see [2,3])

$$\text{Eff} = \frac{|M^{-1}|}{|\Lambda|} \qquad (2.8)$$

where

$$|M^{-1}| = \frac{|A|^2}{|C|} \qquad (2.9)$$

$|\Lambda|$ is given in (2.4), the matices A and C in (2.9) are defined as

$$A = E\begin{bmatrix} \frac{\partial g_1}{\partial p} & \frac{\partial g_1}{\partial r} \\ \\ \frac{\partial g_2}{\partial p} & \frac{\partial g_2}{\partial r} \end{bmatrix}$$

$$C = nE\begin{bmatrix} g_1^2 & g_1 g_2 \\ \\ g_1 g_2 & g_2^2 \end{bmatrix}$$

and g_1, g_2 are from (2.5). After some calculation, A and C turn out to be

$$A = \begin{bmatrix} \frac{r}{p^2} & -\frac{q}{p} \\ \\ -\left(\frac{p}{1-qt}\right)^{r-1} \cdot \frac{r(1-t)}{(1-qt)^2} & \left(\frac{p}{1-qt}\right)^r \log\left(\frac{1-qt}{p}\right) \end{bmatrix} \tag{2.10}$$

$$C = \begin{bmatrix} \frac{rq}{p^2} & -\frac{rqp^{r-1}(1-t)}{(1-qt)^{r+1}} \\ \\ -\frac{rqp^{r-1}(1-t)}{(1-qt)^{r+1}} & \left(\frac{p}{1-qt^2}\right)^r - \left(\frac{p}{1-qt}\right)^{2r} \end{bmatrix} \tag{2.11}$$

The efficiency of the estimation scheme is found, substituting $|A|$, $|C|$ and $|\Lambda|$ into (2.9) and (2.8), as

$$\text{Eff} = \frac{rp^{r+2}}{q} \cdot \frac{\left[\left(\frac{1-qt}{p}\right) \log\left(\frac{1-qt}{p}\right) - \left(\frac{q(1-t)}{p}\right)\right]^2}{\left[\frac{(1-qt)^{2(r+1)}}{(1-qt^2)^r} - p^r(1-qt)^2 - rqp^r(1-t)^2\right]\left[\sum_{n=1}^{\infty} \frac{q^n}{(n+1)} \cdot \frac{r!n!}{(r+n)!}\right]} \tag{2.12}$$

Detailed computations are given in Appendix A. The initial value of r is found using the method of moments $(\tilde{p}, \tilde{r})$, and the value of t used maximizes the efficiency (2.12) when evaluated at $\tilde{p}$, $\tilde{r}$.

III. USAGE AND APPLICATION

In this chapter the procedure to be followed, in order to use the method developed, is given in greater detail and is applied to three sets of data taken from [4] and sixty sets of data generated by computer simulation using various combinations of the parameters.

A. PROCEDURE

The steps of procedure are given below.

1. Using the method of moments, find $\tilde{p}$ and $\tilde{r}$ as starting values. Starting values are given by (3.1) (see [2])

$$\tilde{p} = \bar{x}/s^2$$

$$\tilde{r} = \frac{\bar{x}^2}{s^2 - \bar{x}} \tag{3.1}$$

where $\bar{x}$ is the sample mean and s^2 is the sample variance.

2. Using $\tilde{p}$, $\tilde{r}$ found in step 1, find the value of t, which maximizes efficiency function (2.12), and call it t_o, and also find the efficiency, which is maximum with this pair $\tilde{p}$, $\tilde{r}$ and t_o, and call it EFF1. Tables (4.1) and (4.2) have been prepared for this. Single variable search and golden section search were used by the author to find t_o and EFF1.

3. Using $\tilde{r}$ as a starting value and t_o, found in step 2, find r* as a solution of (2.7), and p* using (2.6). Newton's method was used by the author to find r*.

4. Step 2 can be repeated with p* and r* replacing $\tilde{p}$ and $\tilde{r}$, and using a new t_o (update t_o from Table (4.2) if the current estimated efficiency (Table 4.1) is not sufficiently high). This step involves the recomputations of $\overline{t^x}$ and is seldom needed. The new efficiency using p*, r* and updated t_o is called EFF2.

5. Stop if all the following conditions are met.

$$(1 - \text{EFF1}) \leq e_1 \,, \quad \text{or} \quad |\text{EFF2} - \text{EFF1}| \leq e_2$$

and

$$|p^* - \tilde{p}| \leq e_3,$$

and

$$|r^* - \tilde{r}| \leq e_4 \,.$$

Then p* and r* are the estimators of true parameters p and r, and the estimated efficiency is EFF2.

6. If one or more of the conditions in Step 5 are not met, let

$$\tilde{p} = p^*$$

$$\tilde{r} = r^*$$

$$EFF1 = EFF2$$

and go to step 3 and repeat the steps following until the conditions in step 5 are met. e_1, e_2, e_3 and e_4 are the stopping criteria which can be chosen by the user.

B. EXAMPLES

Examples one through three are the applications of the method to the Cricket score data of Reep, Polard and Benjamin [4]. Example four is based on data generated by computer and contains sixty cases.

Example 1: Applying method developed to the Cowdrey data [4] the following information is obtained.

Sample mean $= 1.692$

Sample variance $= 4.343$

$p^* = 0.325$

$r^* = 0.816$

Optimum $t_o = 0.459$

Efficiency $= 0.996$

Number of Iterations $= 3$

With the same data maximum likelihood estimators $\hat{p}$, $\hat{r}$, and method of moment estimates $\tilde{p}$, $\tilde{r}$ are given in [2] as

$\hat{p} = 0.329$ $\qquad$ $\hat{r} = 0.831$

$\tilde{p} = 0.390$ $\qquad$ $\tilde{r} = 1.080$

Example 2: Example 1 was repeated for Barrington data [4] and following information is obtained.

Sample mean = 2.095

Sample variance = 4.939

$p^* = 0.346$

$r^* = 1.111$

Optimum t_o = 0.538

Efficiency = 0.995

Number of Iterations = 3

$\hat{p} = 0.345$ $\qquad$ $\hat{r} = 1.014$

$\tilde{p} = 0.424$ $\qquad$ $\tilde{r} = 1.543$

Example 3: Like the previous two examples, the following information is obtained by applying the method to the Graveney data [4]:

Sample mean = 1.570

Sample variance = 4.474

$p^* = 0.315$

$r^* = 0.722$

Optimum t_o = 0.430

Efficiency = 0.996

Number of iterations = 2

$\hat{p}$ = 0.317 $\hat{r}$ = 0.729

$\tilde{p}$ = 0.351 $\tilde{r}$ = 0.849

When the first three examples are examined carefully it is noticed that the method developed in this study is almost as efficient as maximum likelihood method. Estimators are pretty close to maximum likelihood estimators and much better than those found by the method of moments.

Example 4: This example is based on the data generated by computer. Sixty cases, each of which has a different sample size and parameters p and r. Four sample sizes, which are 15, 30, 50 and 100; three r values, 0.5, 2.5 and 5; and five p values, 0.05, 0.1, 0.3, 0.5, 0.8 are used. In each case the data is generated for given sample size (n), p and r. Results, which contain case number (case no.), sample size (n), p, r, method of moments estimates ($\tilde{p}$,$\tilde{r}$), optimum u_o at $\tilde{p}$-$\tilde{r}$ (Initial u_o), efficiency at the end of the first iteration (Initial eff.), empirical La Place transform estimates (p*,r*), optimum u_o at p*-r* (Final u_o), efficiency at p*-r*-final u_o (final eff) and number of iterations (No. of itr.), are tabulated in the following pages. Data generated for the sixty cases are given in

Appendix B. The computer program, written in FORTRAN IV, is also given in Appendix C.

When all the cases are examined, it's seen that estimates are not given for cases 2, 3 and 18. Case 3 provides nothing to work with as all samples are zero, yielding $\overline{X} = 0$, $s^2 = 0$. The other two cases have $s^2 < \overline{x}$ which signals trouble because the method of moments estimates $\tilde{r} < 0$, $\tilde{p} > 1$; values which cannot be used to initialize our iteration scheme (3.1) or the maximum likelihood scheme (2.3).

For each of these cases the function f(r) of (2.7) is a non-negative decreasing function of r which tends to $+\infty$ as $r \to 0$ and is asymptotically zero as $r \to \infty$. Thus no root exists. The maximum likelihood approach yields a comparable situation.

CASE NO.	r	p	r	$\tilde{p}$	$\tilde{r}$	Initial u_o	Initial Eff	p*	r*	Final u_o	Final Eff	No. of Itr.
1	15	0.8	5.0	.731	5.081	.2622	.999	.781	6.646	.2162	1.000	3
2	15	0.8	2.5	—	—	—	—	—	—	—	—	—
3	15	0.8	0.5	—	—	—	—	—	—	—	—	—
4	15	0.5	5.0	.468	4.047	.2286	.998	.490	4.425	.2177	.999	3
5	15	0.5	2.5	.539	4.135	.2504	.998	.499	3.524	.2744	.998	4
6	15	0.5	0.5	.459	0.679	1.0334	.997	.540	0.941	.8801	.999	4
7	15	0.3	5.0	.335	5.881	.1187	.998	.324	5.580	.1214	.998	3
8	15	0.3	2.5	.429	4.500	.1919	.998	.429	4.502	.1919	.998	1
9	15	0.3	0.5	.433	1.833	.4550	.991	.319	1.126	.5871	.995	4
10	15	0.1	5.0	.090	4.586	.0452	.988	.074	3.721	.0492	.989	3
11	15	0.1	2.5	.156	3.639	.1106	.989	.135	3.079	.1073	.989	3
12	15	0.1	0.5	.128	.0803	.4937	.983	.111	0.685	.5421	.983	3
13	15	0.05	5.0	.046	4.268	.0265	.991	.052	4.813	.0257	.991	3
14	15	0.05	2.5	.081	4.640	.0415	.988	.067	3.751	.0442	.988	3
15	15	0.05	0.5	.077	0.826	.3529	.964	.050	0.523	.4929	.973	3

CASE NO.	n	p	r	$\tilde{p}$	$\tilde{r}$	Initial u_o	Initial Eff	p*	r*	Final u_o	Final Eff	No. of Itr.
16	30	0.8	5.0	.767	5.587	.2496	1.000	.752	5.151	.2646	1.000	4
17	30	0.8	2.5	.850	5.107	.2929	1.000	.840	4.711	.3114	1.000	4
18	30	0.8	0.5	—	—	—	—	—	—	—	—	—
19	30	0.5	5.0	.449	3.746	.2384	.997	.518	4.938	.2046	.999	5
20	30	0.5	2.5	.592	3.867	.2863	.997	.529	2.995	.3332	.998	4
21	30	0.5	0.5	.559	0.888	.9309	.998	.485	0.659	1.0754	.999	3
22	30	0.3	5.0	.349	6.141	.1176	.998	.337	5.835	.1203	.998	3
23	30	0.3	2.5	.283	2.078	.3019	.994	.311	2.382	.2820	.995	2
24	30	0.3	0.5	.352	0.816	.8083	.996	.343	0.785	.8247	.996	2
25	30	0.1	5.0	.096	4.984	.0446	.994	.103	5.434	.0436	.994	3
26	30	0.1	2.5	.113	2.946	.0963	.986	.106	2.734	.0992	.986	3
27	30	0.1	0.5	.099	0.622	.5681	.983	.092	0.570	.5989	.983	2
28	30	0.05	5.0	.058	6.063	.0218	.995	.059	6.281	.0217	.995	2
29	30	0.05	2.5	.086	4.421	.0465	.986	.068	3.414	.0504	.997	3
30	30	0.05	0.5	.049	0.486	.5324	.975	.048	0.481	.5324	.975	1

CASE NO.	n	p	r	$\tilde{p}$	$\tilde{r}$	Initial u_o	Initial Eff	p*	r*	Final u_o	Final Eff	No. of Itr.
31	50	0.8	5.0	.632	1.858	.5665	.998	.680	2.300	.4973	.999	3
32	50	0.8	2.5	.700	1.916	.5867	.999	.741	2.346	.5164	1.000	3
33	50	0.8	0.5	.731	0.436	1.5736	1.000	.668	0.322	1.7833	1.000	3
34	50	0.5	5.0	.481	4.461	.2125	.999	.508	4.984	.1998	.999	3
35	50	0.5	2.5	.529	3.052	.3275	.998	.493	2.646	.3551	.998	3
36	50	0.5	0.5	.682	0.900	1.0032	.998	.586	0.594	1.2317	.999	4
37	50	0.3	5.0	.252	4.000	.1384	.996	.261	4.198	.1356	.996	2
38	50	0.3	2.5	.239	2.052	.2704	.992	.249	2.167	.2631	.992	2
39	50	0.3	0.5	.332	0.656	.9307	.996	.290	0.539	1.0188	.996	3
40	50	0.1	5.0	.110	5.487	.0459	.995	.113	5.631	.0455	.995	2
41	50	0.1	2.5	.070	1.694	.1271	.979	.098	2.429	.1067	.984	4
42	50	0.1	0.5	.106	0.555	.6596	.985	.100	0.516	.6878	.986	2
43	50	0.05	5.0	.051	4.997	.0244	.991	.048	4.621	.0249	.991	3
44	50	0.05	2.5	.043	2.149	.0598	.978	.050	2.472	.0562	.978	3
45	50	0.05	0.5	.049	0.518	.4948	.974	.046	0.487	.5183	.974	2

CASE NO.	n	p	r	$\tilde{p}$	$\tilde{r}$	Initial u_o	Initial Eff	p*	r*	Final u_o	Final Eff	No. of Itr.
46	100	0.8	5.0	.830	5.758	.2584	1.000	.825	5.550	.2658	1.000	3
47	100	0.8	2.5	.832	2.925	.4636	1.000	.828	2.831	.4740	1.000	2
48	100	0.8	0.5	.874	0.626	1.3707	1.000	.866	0.582	1.4202	1.000	2
49	100	0.5	5.0	.463	4.616	.2083	.998	.513	5.401	.1864	.999	4
50	100	0.5	2.5	.525	2.788	.3540	.998	.565	3.279	.3220	.999	3
51	100	0.5	0.5	.500	0.510	1.2802	.999	.507	0.525	1.2642	.999	2
52	100	0.3	5.0	.302	5.115	.1249	.996	.283	4.670	.1298	.997	3
53	100	0.3	2.5	.282	2.533	.2458	.994	.283	2.544	.2452	.994	2
54	100	0.3	0.5	.261	0.384	1.2397	.997	.263	0.388	1.2341	.997	2
55	100	0.1	5.0	.098	5.356	.0423	.994	.101	5.528	.0419	.994	2
56	100	0.1	2.5	.113	2.847	.1000	.987	.112	2.820	.1004	.987	2
57	100	0.1	0.5	.100	0.553	.6442	.984	.091	0.496	.6871	.985	2
58	100	0.05	5.0	.056	5.623	.0233	.993	.056	5.546	.0234	.993	2
59	100	0.05	2.5	.060	3.020	.0523	.984	.059	2.976	.0526	.984	2
60	100	0.05	0.5	.057	0.657	.3947	.970	.050	0.579	.4336	.971	2

IV. RESULT AND CONCLUSION

In this chapter efficiencies for various p-r pairs were given in Table (4.1). The method developed was used to calculate table entries. Three entries were given for each p-r pair. Entry in the middle is efficiency value for that pair at optimum t_o or u_o. Entries inside parenthesis are also efficiencies and found as follows:

Let $u_o = - \log t_o$

If we miss u_o by ±50 percent, what would the efficiency be? Efficiency inside the parenthesis above the optimum efficiency answers the question for $u_1 = 0.5\, u_o$, and other one below the optimum answers for $u_2 \approx 1.5\, u_o$.

The change $u = -\log t$ (or rather its inverse $t = e^{-u}$) changes the probability generating $\overline{t^x}$ function into the empirical La Place transform. It is seen that a 50 percent error in its argument u results in only a minor degradation of efficiency, whereas a 50 percent error in t would have disasterous effects. In all of the examples, the initial u_o was well within this 50 percent range, the worse case being 38 percent.

After examining Table (4.1) one can conclude that efficiency is monotonically increasing with increasing p and increasing r for r greater than one and minimum efficiency at optimum t_o or u_o is greater than 0.95.

Following Table (4.1) optimum t_o and u_o (in parenthesis) were tabulated in Table (4.2). One may observe that t_o decreases with p and increases with r, whereas u_o behaves oppositely.

After Table (4.2) efficiency contours were graphed in p-r plane (Figure 4.1), in mean-r plane (Figure 4.2) and in mean-exp(r) plane (Figure 4.3). Compared with the Figure in [5] Figure (4.3) always gives better efficiency values than all the other alternatives.

TABLE 4.1 ASYMPTOTIC EFFICIENCY OF p*, r*

	0.05	0.06	0.07	0.08	0.09	0.1	0.2
	(.961)	(.964)	(.967)	(.969)	(.971)	(.972)	(.982)
0.5	.974	.978	.981	.983	.985	.986	.994
	(.933)	(.935)	(.937)	(.939)	(.941)	(.942)	(.952)
	(.943)	(.947)	(.950)	(.953)	(.955)	(.957)	(.970)
1.0	.963	.967	.970	.973	.976	.978	.989
	(.912)	(.916)	(.919)	(.921)	(.924)	(.926)	(.940)
	(.943)	(.946)	(.948)	(.950)	(.952)	(.954)	(.966)
1.5	.967	.970	.972	.974	.976	.978	.989
	(.912)	(.915)	(.918)	(.920)	(.922)	(.924)	(.939)
	(.947)	(.949)	(.950)	(.952)	(.953)	(.955)	(.965)
2.0	.973	.975	.977	.978	.980	.981	.989
	(.919)	(.921)	(.923)	(.925)	(.927)	(.928)	(.942)
	(.951)	(.952)	(.954)	(.955)	(.956)	(.957)	(.966)
2.5	.979	.980	.981	.982	.983	.984	.991
	(.926)	(.928)	(.929)	(.931)	(.932)	(.934)	(.945)

TABLE 4.1. ASYMPTOTIC EFFICIENCY OF p*, r* (Cont'd)

	0.05	0.06	0.07	0.08	0.09	0.1	0.2
	(.955)	(.956)	(.957)	(.958)	(.959)	(.959)	(.967)
3.0	.983	.984	.985	.985	.986	.987	.992
	(.933)	(.934)	(.935)	(.937)	(.938	(.939)	(.949)
	(.958)	(.959)	(.960)	(.960)	(.961)	(.962)	(.968)
3.5	.986	.987	.987	.988	.989	.989	.993
	(.939)	(.940)	(.941)	(.942)	(.943)	(.932)	(.877)
	(.961)	(.961)	(.962)	(.963)	(.963)	(.964)	(.970)
4.0	.989	.989	.990	.990	.990	.991	.994
	(.944)	(.945)	(.946)	(.947)	(.998)	(.948)	(.956)
	(.963)	(.964)	(.964)	(.965)	(.965)	(.966)	(.971)
4.5	.990	.991	.991	.992	.992	.992	.995
	(.948)	(.949)	(.950)	(.951)	(.951)	(.952)	(.959)
	(.965)	(.966)	(.966)	(.967)	(.967)	(.968)	(.973)
5.	.992	.992	.992	.993	.993	.993	.996
	(.952)	(.953)	(.954)	(.954)	(.955)	(.956)	(.962)

TABLE 4.1. ASYMPTOTIC EFFICIENCY OF p*, r* (Cont'd)

	0.05	0.06	0.07	0.08	0.09	0.1	0.2
	(.969)	(.969)	(.970)	(.970)	(.970)	(.971)	(.975)
6.	.994	.994	.994	.995	.995	.995	.997
	(.958)	(.959)	(.959)	(.960)	(.961)	(.961)	(.967)
	(.972)	(.972)	(.972)	(.973)	(.973)	(.973)	(.977)
7.	.995	.996	.996	.996	.996	.996	.997
	(.963)	(.964)	(.964)	(.965)	(.965)	(.966)	(.970)
	(.974)	(.974)	(.975)	(.975)	(.975)	(.976)	(.979)
8.	.996	.996	.997	.997	.997	.997	.998
	(.967)	(.997)	(.968)	(.968)	(.969)	(.969)	(.973)
	(.976)	(.976)	(.977)	(.977)	(.977)	(.977)	(.980)
9.	.997	.997	.997	.997	.997	.997	.998
	(.970)	(.970)	(.971)	(.971)	(.971)	(.972)	(.975)
	(.978)	(.978)	(.978)	(.978)	(.979)	(.979)	(.982)
10.	.998	.998	.998	.998	.998	.998	.998
	(.973)	(.973)	(.973)	(.974)	(.974)	(.974)	(.978)

TABLE 4.1. ASYMPTOTIC EFFICIENCY OF p*, r* (Cont'd)

	0.05	0.06	0.07	0.08	0.09	0.1	0.2
	(.979)	(.979)	(.980)	(.980)	(.980)	(.980)	(.983)
11.	.998	.998	.998	.998	.998	.998	.999
	(.975)	(.975)	(.975)	(.976)	(.976)	(.976)	(.976)
	(.980)	(.981)	(.981)	(.981)	(.981)	(.982)	(.984)
12.	.998	.998	.998	.998	.998	.998	.999
	(.978)	(.977)	(.977)	(.977)	(.978)	(.978)	(.981)
	(.982)	(.982)	(.983)	(.983)	(.983)	(.984)	(.986)
13.	.998	.999	.999	.999	.999	.999	.999
	(.978)	(.980)	(.980)	(.980)	(.981)	(.981)	(.983)
	(.983)	(.983)	(.983)	(.983)	(.983)	(.984)	(.986)
14.	.999	.999	.999	.999	.999	.999	.999
	(.980)	(.980)	(.980)	(.980)	(.981)	(.981)	(.983)
	(.983)	(.984)	(.984)	(.984)	(.984)	(.984)	(.986)
15.	.999	.999	.999	.999	.999	.999	.999
	(.981)	(.981)	(.981)	(.982)	(.982)	(.982)	(.984)

TABLE 4.1. ASYMPTOTIC EFFICIENCY OF p*, r* (Cont'd)

	0.05	0.06	0.07	0.08	0.09	0.1	0.2
	(.984)	(.984)	(.985)	(.985)	(.985)	(.985)	(.987)
16	.999	.999	.999	.999	.999	.999	.999
	(.982)	(.982)	(.982)	(.983)	(.983)	(.983)	(.985)
	(.985)	(.985)	(.985)	(.986)	(.986)	(.986)	(.988)
17	.999	.999	.999	.999	.999	.999	.999
	(.983)	(.983)	(.983)	(.983)	(.984)	(.984)	(.986)
	(.986)	(.986)	(.986)	(.986)	(.986)	(.987)	(.988)
18	.999	.999	.999	.999	.999	.999	.999
	(.984)	(.984)	(.984)	(.984)	(.985)	(.985)	(.987)
	(.986)	(.986)	(.987)	(.987)	(.987)	(.997)	(.989)
19	.999	.999	.999	.999	.999	.999	.999
	(.985)	(.985)	(.985)	(.985)	(.985)	(.985)	(.987)

TABLE 4.1. ASYMPTOTIC EFFICIENCY OF p*, r* (Cont'd)

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	(.987)	(.990)	(.993)	(.995)	(.996)	(.998)	(.999)
0.5	.997	.998	.999	1.000	1.000	1.000	1.000
	(.959)	(.966)	(.972)	(.978)	(.984)	(.990)	(.995)
	(.977)	(.983)	(.987)	(.990)	(.993)	(.996)	(.998)
1.0	.994	.997	.998	.999	1.000	1.000	1.000
	(.951)	(.959)	(.967)	(.974)	(.981)	(.988)	(.994)
	(.973)	(.979)	(.984)	(.988)	(.991)	(.995)	(.997)
1.5	.994	.996	.998	.999	1.000	1.000	1.00
	(.950)	(.959)	(.967)	(.974)	(.981)	(.988)	(.994)
	(.972)	(.978)	(.983)	(.987)	(.990)	(.994)	(.997)
2.0	.994	.996	.998	.999	.999	1.000	1.000
	(.952)	(.960)	(.968)	(.975)	(.982)	(.988)	(.994)
	(.972)	(.978)	(.982)	(.986)	(.990)	(.994)	(.997)
2.5	.994	.997	.998	.999	.999	1.000	1.000
	(.955)	(.963)	(.970)	(.977)	(.983)	(.989)	(.995)

TABLE 4.1. ASYMPTOTIC EFFICIENCY OF p*, r* (Cont'd)

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	(.973)	(.978)	(.982)	(.986)	(.990)	(.994)	(.997)
3.0	.995	.997	.998	.999	1.000	1.000	1.000
	(.958)	(.965)	(.972)	(.978)	(.984)	(.990)	(.995)
	(.974)	(.978)	(.983)	(.987)	(.990)	(.994)	(.997)
3.5	.996	.997	.998	.999	1.000	1.000	1.000
	(.961)	(.967)	(.974)	(.979)	(.985)	(.990)	(.995)
	(.975)	(.979)	(.983)	(.987)	(.990)	(.994)	(.997)
4.0	.998	.998	.999	.999	1.000	1.000	1.000
	(.963)	(.970)	(.975)	(.981)	(.986)	(.991)	(.995)
	(.976)	(.980)	(.984)	(.987)	(.991)	(.994)	(.997)
4.5	.997	.998	.999	.999	1.000	1.000	1.000
	(.966)	(.971)	(.977)	(.982)	(.987)	(.991)	(.996)
	(.977)	(.981)	(.984)	(.988)	(.991)	(.994)	(.997)
5.0	.997	.998	.999	.999	1.000	1.000	1.000
	(.968)	(.973)	(.978)	(.983)	(.987)	(.992)	(.996)

TABLE 4.1. ASYMPTOTIC EFFICIENCY OF p*, r* (Cont'd)

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	(.979)	(.982)	(.985)	(.989)	(.992)	(.994)	(.997)
6.0	.998	.998	.999	.999	1.000	1.000	1.000
	(.972)	(.976)	(.981)	(.985)	(.989)	(.993)	(.996)
	(.980)	(.983)	(.986)	(.989)	(,992)	(.995)	(.997)
7.0	.998	.999	.999	1.000	1.000	1.000	1.000
	(.974)	(.979)	(.982)	(.986)	(.990)	(.993)	(.997)
	(.982)	(.985)	(.987)	(.990)	(.993)	(.995)	(.998)
8.0	.998	.000	.000	1.000	1.000	1.000	1.000
	(.977)	(.981)	(.984)	(9.87)	(.991)	(.994)	(.997)
	(.983)	(.986)	(.988)	(.991)	(.993)	(.995)	(.998)
9.0	.999	.999	.999	1.000	1.000	1.000	1.000
	(.979)	(.982)	(.985)	(.989)	(.992)	(.994)	(.997)
	(.984)	(.987)	(.989)	(.991)	(.993)	(.996)	(.998)
10.0	.999	.999	1.000	1.000	1.000	1.000	1.000
	(.981)	(.984)	(.987)	(.989)	(.992)	(.995)	(.997)

TABLE 4.1. ASYMPTOTIC EFFICIENCY OF p^*, r^* (Cont'd)

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	(.985)	(.987)	(.990)	(.992)	(.994)	(.996)	(.998)
11.0	.999	.999	1.000	1.000	1.000	1.000	1.000
	(.982)	(.985)	(.988)	(.990)	(.993)	(.995)	(.998)
	(.986)	(.988)	(.990)	(.992)	(.994)	(.996)	(.998)
12.0	.999	.999	1.000	1.000	1.000	1.000	1.000
	(.983)	(.986)	(.988)	(.991)	(.993)	(.996)	(.998)
	(.987)	(.989)	(.991)	(.993)	(.995)	(.996)	(.998)
13.0	.999	.999	1.000	1.000	1.000	1.000	1.000
	(.984)	(.987)	(.989)	(.991)	(.994)	(.996)	(.998)
	(.987)	(.989)	(.991)	(.993)	(.995)	(.997)	(.998)
14.0	.999	1.000	1.000	1.000	1.000	1.000	1.000
	(.985)	(.988)	(.990)	(.992)	(.994)	(.996)	(.998)
	(.988)	(.990)	(.992)	(.993)	(.995)	(.997)	(.998)
15.0	.999	1.000	1.000	1.000	1.000	1.000	1.000
	(.986)	(.988)	(.990)	(.992)	(.994)	(.996)	(.998)

TABLE 4.1. ASYMPTOTIC EFFICIENCY OF p*, r* (Cont'd)

	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	(.989)	(.990)	(.992)	(.994)	(.995)	(.997)	(.998)
16.0	.999	1.000	1.000	1.000	1.000	1.000	1.000
	(.987)	(.989)	(.991)	(.993)	(.995)	(.996)	(.998)
	(.989)	(.991)	(.992)	(.994)	(.995)	(.997)	(.999)
17.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(.988)	(.990)	(.991)	(.993)	(.995)	(.997)	(.998)
	(.990)	(.991)	(.993)	(.994)	(.996)	(.997)	(.999)
18.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(.988)	(.990)	(.992)	(.993)	(.995)	(.997)	(.998)
	(.990)	(.991)	(.993)	(.994)	(.996)	(.997)	(.999)
19.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	(.989)	(.991)	(.992)	(.994)	(.995)	(.997)	(.998)

TABLE 4.2 OPTIMUM t_o AND u_o FOR $p^* - r^*$

	0.05	0.06	0.07	0.08	0.09	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.5	.594	.567	.544	.524	.507	.491	.391	.337	.300	.274	.253	.236	.222	.210
	(.520)	(.567)	(.608)	(.646)	(.680)	(.711)	(.938)	(1.088)	(1.202)	(1.295)	(1.374)	(1.443)	(1.504)	(1.559)
1.0	.817	.794	.772	.753	.735	.719	.603	.532	.482	.444	.413	.388	.367	.349
	(.202)	(.231)	(.258)	(.284)	(.308)	(.330)	(.506)	(.631)	(.730)	(.813)	(.883)	(.946)	(1.002)	(1.052)
1.5	.894	.877	.861	.847	.833	.820	.717	.648	.596	.554	.521	.492	.468	.447
	(.112)	(.131)	(.149)	(.166)	(.183)	(.199)	(.332)	(.434)	(.518)	(.590)	(.653)	(.708)	(.759)	(.805)
2.0	.928	.915	.903	.892	.881	.871	.785	.721	.671	.630	.596	.567	.542	.520
	(.075)	(.089)	(.102)	(.114)	(.127)	(.139)	(.243)	(.327)	(.399)	(.461)	(.517)	(.567)	(.612)	(.654)
2.5	.946	.936	.927	.918	.909	.900	.827	.770	.724	.685	.652	.624	.598	.576
	(.056)	(.066)	(.076)	(.086)	(.096)	(.105)	(.190)	(.261)	(.323)	(.378)	(.427)	(.472)	(.514)	(.552)
3.0	.957	.949	.941	.934	.926	.919	.856	.805	.763	.727	.695	.667	.642	.620
	(.044)	(.053)	(.061)	(.069)	(.077)	(.084)	(.155)	(.216)	(.271)	(.319)	(.364)	(.405)	(.443)	(.478)
3.5	.964	.958	.951	.945	.938	.932	.877	.832	.792	.758	.728	.702	.678	.656
	(.036)	(.043)	(.050)	(.057)	(.064)	(.070)	(.131)	(.184)	(.233)	(.276)	(.317)	(.354)	(.389)	(.421)
4.0	.969	.964	.958	.953	.947	.942	.893	.852	.816	.784	.756	.730	.707	.686
	(.031)	(.037)	(.043)	(.049)	(.054)	(.060)	(.113)	(.161)	(.204)	(.244)	(.280)	(.315)	(.347)	(.377)

TABLE 4.2 OPTIMUM t_o AND u_o FOR $p^* = r^*$ (Cont'd)

	0.05	0.06	0.07	0.08	0.09	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
4.5	.973	.968	.963	.959	.954	.949	.905	.868	.834	.804	.778	.753	.731	.711
	(.027)	(.032)	9.037)	(.042)	(.047)	(.052)	(.099)	(.142)	(.181)	(.218)	(.251)	(.283)	(.313)	(.341)
5.0	.976	.972	.968	.963	.959	.955	.915	.880	.849	.822	.796	.773	.752	.732
	(.024)	(.028)	(.033)	(.037)	(.042)	(.046)	(.089)	(.127)	(.163)	(.197)	(.228)	(.257)	(.285)	(.311)
6.0	.981	.977	.974	.970	.966	.963	.930	.900	.873	.848	.826	.805	.785	.767
	(.019)	(.023)	(.027)	(.030)	(.034)	(.038)	(.073)	(.105)	(.136)	(.165)	(.192)	(.217)	(.242)	(.265)
7.0	.984	.981	.978	.975	.972	.969	.940	.914	.890	.868	.848	.828	.810	.794
	(.016)	(.019)	(.023)	(.026)	(.029)	(.032)	(.062)	(.090)	(.116)	(.141)	(.165)	(.188)	(.210)	(.231)
8.0	.986	.983	.981	.978	.975	.973	.948	.925	.903	.883	.865	.847	.830	.815
	(.014)	(.017)	(.019)	(.022)	(.025)	(.028)	(.054)	(.078)	(.102)	(.124)	(.146)	(.166)	(.186)	(.205)
9.0	.988	.985	.983	.981	.978	.976	.954	.933	.914	.895	.878	.862	.847	.832
	(.012)	(.015)	(.017)	(.019)	(.022)	(.024)	(.047)	(.069)	(.090)	(.110)	(.130)	(.148)	(.166)	(.184)
10.0	.989	.987	.985	.983	.981	.979	.959	.940	.922	.905	.889	.874	.860	.846
	(.011)	(.013)	(.015)	(.017)	(.019)	(.022)	(.042)	(.062)	(.081)	(.100)	(.117)	(.134)	(.151)	(.167)
11.0	.990	.988	.986	.984	.983	.981	.962	.945	.929	.913	.899	.885	.871	.858
	(.010)	(.012)	(.014)	(.016)	(.018)	(.020)	(.038)	(.056)	(.074)	(.091)	(.107)	(.123)	(.138)	(.153)

TABLE 4.2 OPTIMUM t_o AND u_o FOR p* - r* (Cont'd)

	0.05	0.06	0.07	0.08	0.09	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
12.0	.991	.989	.988	.986	.084	.082	.966	.950	.935	.920	.907	.893	.881	.869
	(.009)	(.011)	(.013)	(.014)	(.016)	(.018)	(.035)	(.051)	(.068)	(.083)	(.098)	(.113)	(.127)	(.141)
13.0	.992	.990	.989	.987	.985	.984	.968	.954	.940	.926	.913	.901	.889	.878
	(.008)	(.010)	(.011)	(.013)	(.015)	(.016)	(.032)	(.047)	(.062)	(.077)	(.091)	(.104)	(.118)	(.130)
14.0	.992	.991	.989	.988	.986	.985	.971	.957	.944	.931	.919	.907	.896	.885
	(.008)	(.009)	(.011)	(.012)	(.014)	(.015)	(.030)	(.044)	(.058)	(.071)	(.084)	(.097)	(.110)	(.112)
15.0	.993	.992	.990	.989	.987	.986	.973	.960	.948	.936	.924	.913	.903	.892
	(.007)	(.008)	(.010)	(.011)	(.013)	(.014)	(.028)	(.041)	(.054)	(.067)	(.079)	(.091)	(.103)	(.114)
16.0	.993	.992	.991	.990	.988	.987	.974	.962	.951	.940	.929	.918	.908	.898
	(.007)	(.008)	(.009)	(.011)	(.012)	(.013)	(.026)	(.038)	(.051)	(.062)	(.074)	(.085)	(.096)	(.107)
17.0	.994	.993	.991	.990	.989	.988	.976	.965	.954	.943	.933	.923	.913	.904
	(.006)	(.007)	(.009)	(.011)	(.012)	(.024)	(.036)	(.048)	(.059)	(.070)	(.080)	(.091)	(.091)	(.101)
18.0	.994	.993	.992	.991	.990	.988	.977	.967	.956	.946	.936	.927	.918	.909
	(.006)	(.007)	(.008)	(.009)	(.010)	(.012)	(.023)	(.034)	(.045)	(.055)	(.066)	(.076)	(.086)	(.096)
19.0	.994	.993	.992	.991	.990	.989	.979	.968	.958	.949	.939	.930	.922	.913
	(.006)	(.007)	(.008)	(.009)	(.010)	(.011)	(.022)	(.032)	(.042)	(.053)	(.062)	(.072)	(.082)	(.091)

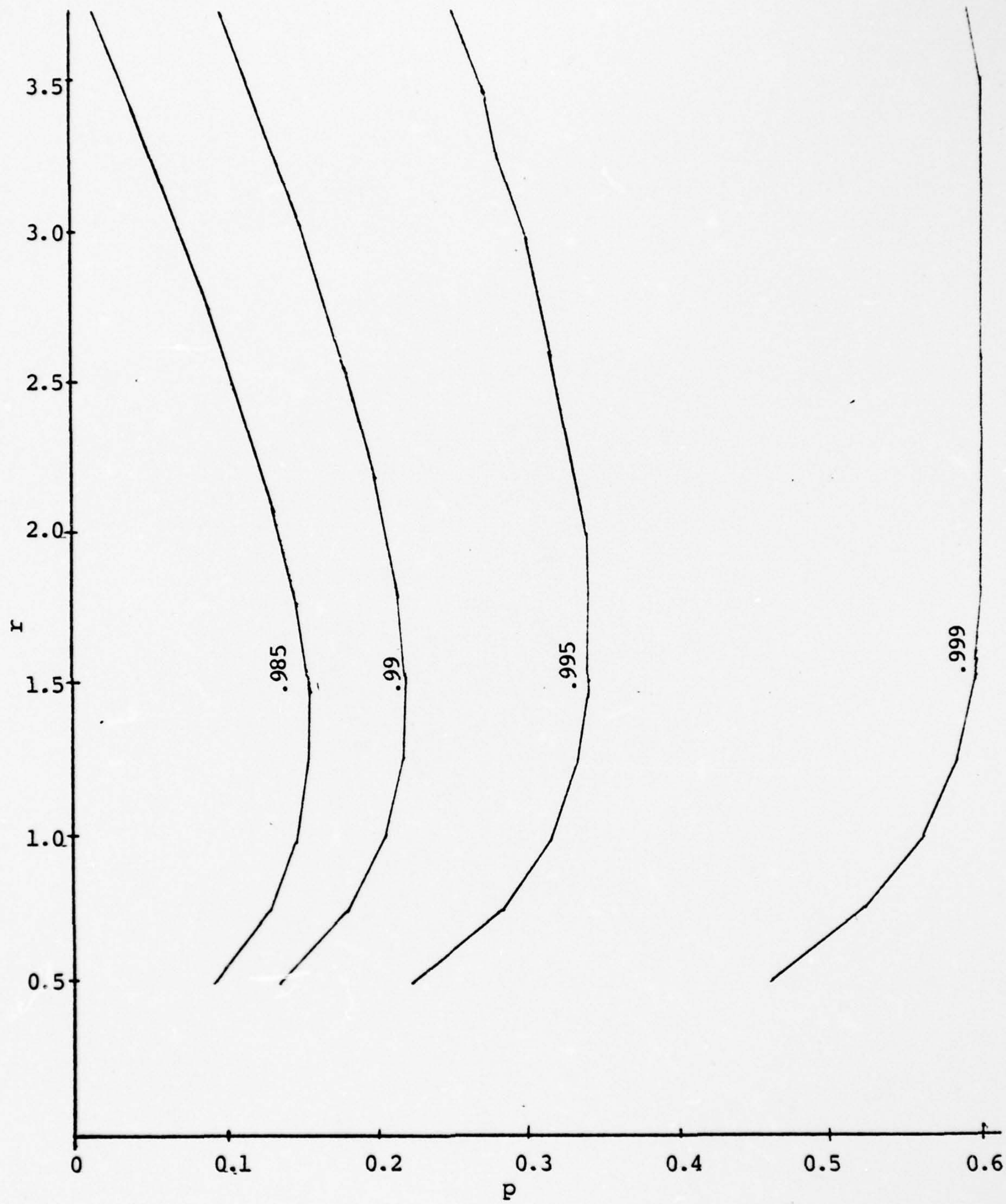

Figure 4.1 Efficiency contours in p-r plane

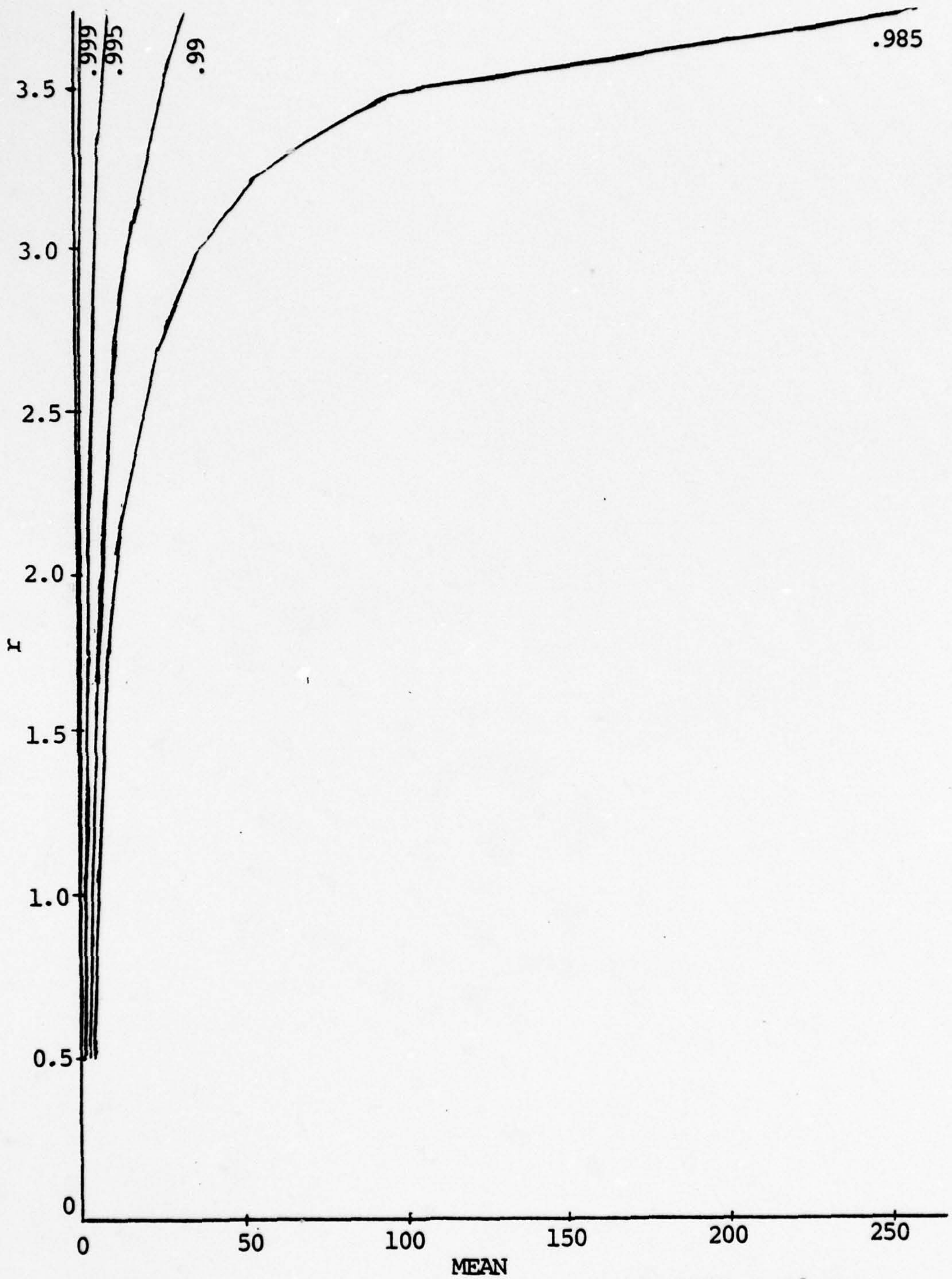

Figure 4.2 Efficiency contours in MEAN-r plane

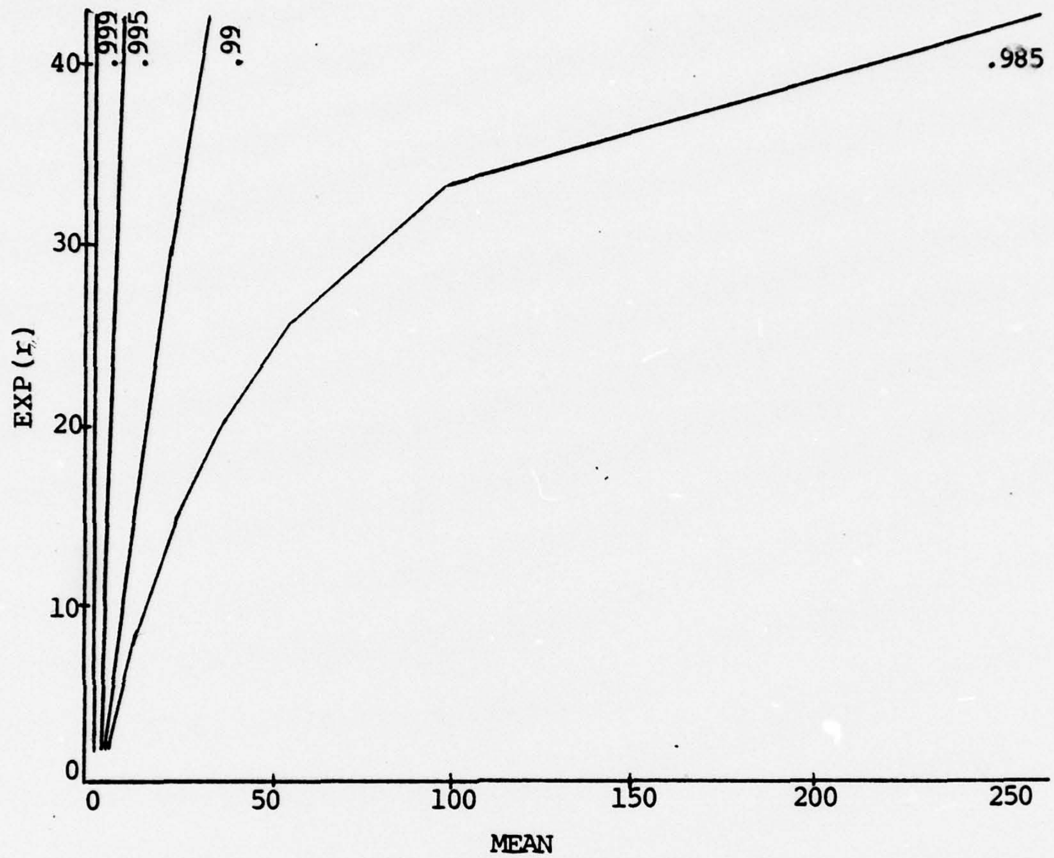

Figure 4.3 Efficiency contours in MEAN-EXP(r) plane

APPENDIX A

DETAILED COMPUTATIONS

The two estimating equations g_1 and g_2 given in chapter II are

$$g_1 = \bar{x} - \frac{rq}{p} = 0$$

$$g_2 = \overline{t^x} - \left(\frac{p}{1-qt}\right)^r = 0$$

Matrix A's entries were given as follows:

$$A_{11} = E\left(\frac{\partial g_1}{\partial p}\right) = E\left(\frac{r}{p^2}\right) = \frac{r}{p^2}$$

$$A_{12} = E\left(\frac{\partial g_1}{\partial r}\right) = E\left(-\frac{q}{p}\right) = -\frac{q}{p}$$

$$A_{21} = E\left(\frac{\partial g_2}{\partial p}\right) = E\left[-\left(\frac{p}{1-qt}\right)^{r-1} \cdot \frac{r(1-t)}{(1-qt)^2}\right]$$

$$= -\left(\frac{p}{1-qt}\right)^{r-1} \cdot \frac{r(1-t)}{(1-qt)^2}$$

$$A_{22} = E\left(\frac{\partial g_2}{\partial r}\right) = E\left[\ln\left(\frac{1-qt}{p}\right)\left(\frac{p}{1-qt}\right)^r\right]$$

$$= \ln\left(\frac{1-qt}{p}\right)\left(\frac{p}{1-qt}\right)^r$$

Matrix C's entries were given as follows:

$$C_{11} = n\,E(g_1^2) = n\,E[(\overline{x} - \frac{rq}{p})^2]$$

$$= n\,\mathrm{Var}(\overline{x}) = n\,\mathrm{Var}(\frac{1}{n} \sum_{i=1}^{n} x_i)$$

$$= n \cdot \frac{1}{n^2}\,\mathrm{Var}(\sum_{i=1}^{n} x_i) = \mathrm{Var}(x)$$

$$= \frac{rq}{p^2}$$

$$C_{12} = n\,E[g_1 \cdot g_2]$$

$$= n\,E[(\overline{x} - \frac{rq}{p})(\overline{t^x} - (\frac{p}{1-qt})^r)]$$

$$= n\,\mathrm{Cov}(\overline{x}, \overline{t^x}) = n\,\mathrm{Cov}(\frac{1}{n} \sum_{i=1}^{n} x_i, \overline{t^x})$$

$$= \mathrm{Cov}(x_i, \overline{t^x})$$

$$= \sum_{i=1}^{n} \mathrm{Cov}(x_i, \overline{t^x}) = \sum_{i=1}^{n} \mathrm{Cov}(x_i, \frac{1}{n} \sum_{j=1}^{n} t^{x_j})$$

$$= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{Cov}(x_i, t^{x_j})$$

Since $\mathrm{Cov}(x_i, t^{x_j}) = 0$ for $i \neq j$ because of independence, it follows that

$$C_{12} = C_{21} = \frac{1}{n} \sum_{i=1}^{n} \mathrm{Cov}(x_i, t^{x_i})$$

$$= \mathrm{Cov}(x, t^x) = E(x \cdot t^x) - E(x)E(t^x)$$

One can calculate $E(x \cdot t^x)$ using the following relation.

$$G(t) = E(t^x) \qquad G'(t) = E(xt^{x-1})$$

Then

$$E(xt^x) = tG'(t)$$

where

$$G(t) = \left(\frac{p}{1-qt}\right)^r$$

taking the derivative and multiplying by t, $E(xt^x)$ is found as

$$E(xt^x) = \frac{trqp^r}{(1-qt)^{r+1}}$$

Then it yields

$$C_{12} = C_{21} = -\frac{rqp^{r-1}(1-t)}{(1-qt)^{r+1}}$$

$$C_{22} = n\,E(g_2^2) = E[(\overline{t^x} - (\frac{p}{1-qt})^r)^2]$$

$$= n\,\mathrm{Var}(\overline{t^x}) = n\,\mathrm{Var}(\frac{1}{n}\sum_{i=1}^{n} t^{x_i})$$

$$= \frac{1}{n}\mathrm{Var}(\sum_{i=1}^{n} t^{x_i})$$

$$= \mathrm{Var}(t^x) = E(t^{2x}) - [E(t^x)]^2$$

$$= G(t^2) - [G(t)]^2$$

$$= (\frac{p}{1-qt^2})^r - (\frac{p}{1-qt})^{2r}$$

As the reader will recall, $|\Lambda|$ was given by (2.4) and can be written as follows.

$$|\Lambda| = \frac{1}{p^2}[\sum_{n=1}^{m} \frac{q^n}{n+1}\cdot\frac{r!n!}{(n+r)!} + \sum_{n=m+1}^{\infty} \frac{q^n}{n+1}\,\frac{r!n!}{(n+r)!}]$$

Call the summation from $n = m+1$ to ∞ the remainder. The remainder can be adjusted such that it will be smaller than any specified small value (ε).

$$\sum_{n=m+1}^{\infty} \frac{q^n}{n+1} \frac{r!n!}{(n+r)!} \leq \sum_{n=m+1}^{\infty} \frac{q^n}{n+1}$$

Because $\frac{r!n!}{(n+r)!} \leq 1$ (reciprocal of binomial coefficient),

$$\sum_{n=m+1}^{\infty} \frac{q^n}{n+1} \leq \frac{q^{m+1}}{m+2} \sum_{n=0}^{\infty} q^n = \frac{q^{m+1}}{(m+2)p} \leq \varepsilon$$

A value of m can be found such that

$$\frac{q^{m+1}}{(m+2)p} \leq \varepsilon$$

holds. Then this m guarantees the remainder will be smaller than ε, and the efficiency equation of (2.12) becomes

$$\text{Eff} = \frac{rp^{r+2}}{q} \cdot \frac{\left[\frac{1-qt}{p} \ln\left(\frac{1-qt}{p}\right) - \frac{q(1-t)}{p}\right]^2}{\left[\frac{(1-qt)^{2(r+1)}}{(1+qt^2)^r} - p^r(1-qt)^2 - rqp^r(1-t)^2\right]\left(\sum_{n=1}^{m} \frac{q^n}{n+1} \frac{r!n!}{(r+n)!}\right)}$$

APPENDIX B

SAMPLES GENERATED BY COMPUTER

Samples generated by computer for sixty cases used in chapter three, example four, are given in the following pages case by case.

See subroutine GNRNB of computer program-3 of Appendix C for the simulation procedure, which was taken from [6].

CASE 1

VAL.	FRQ.
0	2
1	7
2	1
3	3
5	2

CASE 2

VAL.	FRQ.
0	8
1	5
2	2

CASE 3

VAL.	FRQ.
0	15

CASE 4

VAL.	FRQ.
0	1
1	1
2	1
3	4
4	1
5	3
7	2
9	1
12	1

CASE 5

VAL.	FRQ.
0	2
1	1
2	4
3	1
4	2
5	1
7	4

CASE 6

VAL.	FRQ.
0	8
1	5
2	1
5	1

CASE 7

VAL.	FRQ.
3	2
5	1
8	1
9	1
10	2
11	1
12	1
13	1
14	1
15	1
19	1
21	1
22	1

CASE 8

VAL.	FRQ.
1	1
2	2
3	3
5	1
6	1
7	1
8	3
9	1
11	1
14	1

CASE 9

VAL.	FRQ.
0	5
1	2
2	2
4	2
5	2
6	2

CASE 10

VAL.	FRQ.
11	1
14	1
21	1
30	1
33	2
39	1
40	1
57	1

CASE 10
(Cont'd)

VAL.	FRQ.
63	1
64	1
71	1
73	1
74	1
76	1

CASE 11

VAL.	FRQ.
2	1
7	1
8	1
9	1
11	1
17	1
18	1
19	1
22	2
25	1
27	1
30	1
37	1
41	1

CASE 12

VAL.	FRQ.
0	4
1	2
4	1
5	1
6	3
7	1
10	1
11	1
25	1

CASE 13

VAL.	FRQ.
33	1
49	1
52	1
59	1
60	1
63	1
64	1

CASE 13
(Cont'd)

VAL.	FRQ.
68	1
70	1
88	1
120	1
131	1
139	1
153	1
176	1

CASE 14

VAL.	FRQ.
12	1
15	1
24	1
30	1
40	1
42	1
50	1
52	1
59	1
62	1
70	1
71	1
78	1
85	1
96	1

CASE 15

VAL.	FRQ.
0	3
1	2
2	1
3	1
5	1
9	2
15	1
18	1
22	1
30	1
34	1

CASE 16

VAL.	FRQ.
0	8
1	7
2	6
3	6
4	1
5	2

CASE 17

VAL.	FRQ.
0	14
1	8
2	5
3	3

CASE 18

VAL.	FRQ.
0	27
1	3

CASE 19

VAL.	FRQ.
1	2
2	6
3	7
4	3
5	4
6	3
7	1
10	1
11	2
14	1

CASE 20

VAL.	FRQ.
0	5
1	8
2	2
3	2
4	7
5	4
7	2

CASE 21

VAL.	FRQ.
0	19
1	5
2	3
3	2
4	1

CASE 22

VAL.	FRQ.
1	1
3	2
6	1
7	3
8	4
9	2
11	5
12	1
13	1
14	2
15	1
16	1
17	1
19	2
20	2
26	1

CASE 23

VAL.	FRQ.
0	3
1	2
2	2
3	3
4	4
5	5
6	4
7	1
8	2
10	2
15	1
20	1

CASE 24

VAL.	FRQ.
0	12
1	10
2	2
3	1
4	1
5	2
7	2

CASE 25

VAL.	FRQ.
14	1
23	1
28	3
29	1
30	1
31	1
32	1
33	4
34	1
36	1
37	2
47	1
51	1
53	1
58	1
60	1
67	1
68	1
71	1
73	1
77	1
81	1
88	1
102	1

CASE 26

VAL.	FRQ.
0	1
5	1
6	1
8	1
9	1
10	1
12	1
15	3
16	2
17	1
20	2
21	1
23	1
24	1
27	2
29	1
31	1
32	1
33	1
35	1
47	1
49	1
53	1
61	1

CASE 27

VAL.	FRQ.
0	8
1	4
2	1
3	3
4	3
5	2
7	1
8	1
9	2
11	1
15	1
18	1
26	1
29	1

CASE 28

VAL.	FRQ.
42	1
46	1
52	1
55	1
57	1
58	1
59	1
60	1
65	1
72	1
73	1
79	1
81	1
91	1
92	1
98	1
100	1
106	1
112	1
118	1
120	1
121	1
124	1
129	1
132	1
141	1
148	1
167	1
185	1
197	1

CASE 29

VAL.	FRQ.
7	1
12	1
15	1
18	1
21	3
22	1
26	1
30	1
33	1
40	1
49	1
52	1
54	4
55	2
57	1
58	1
65	1
66	1
68	1
71	2
74	1
86	1
95	1

CASE 30

VAL.	FRQ.
0	6
1	5
2	2
3	2
4	1
5	2
6	1
8	1
9	1
10	1
12	2
15	1
21	1
25	1
38	1
47	1
52	1

CASE 31

VAL.	FRQ.
0	21
1	13
2	11
3	4
7	1

CASE 32

VAL.	FRQ.
0	24
1	17
2	6
3	1
4	1
5	1

CASE 33

VAL.	FRQ.
0	44
1	4
2	2

CASE 34

VAL.	FRQ.
1	8
2	6
3	3
4	7
5	10
6	4
7	2
8	4
9	4
11	1
17	1

CASE 35

VAL.	FRQ.
0	9
1	9
2	8
3	8
4	5
5	6
6	1
7	2
8	1
9	1

CASE 36

VAL.	FRQ.
0	37
1	6
2	6
3	1

CASE 37

VAL.	FRQ.
2	1
3	3
4	3
5	2
6	2
7	2
8	5
9	3
10	3
11	2
12	6
13	1
14	2
15	4
17	4
20	1
22	2
23	1
25	1
29	1
33	1

CASE 38

VAL.	FRQ.
0	1
1	7
2	4
3	3
4	5
5	6
6	4
7	6
8	2
10	3
11	2
14	1
15	2
16	2
17	1
25	1

CASE 39

VAL.	FRQ.
0	27
1	5
2	8
3	4
4	3
5	1
6	1
10	1

CASE 40

VAL.	FRQ.
8	1
16	1
17	1
22	1
26	3
27	4
28	1
30	1
31	1
32	3
33	3
35	2
36	2
37	1
40	2
41	1
44	2
45	1
47	1
48	1
49	1
54	2
55	1
58	1
60	1
61	1
62	1
68	1
70	1
71	2
74	1
75	1
87	1
91	1
93	1

CASE 41

VAL.	FRQ.
3	1
4	2
8	3
9	3
10	3
11	1
12	1
13	1
14	4
15	4
16	1
17	2
18	3
19	1
20	2
21	1
22	1
23	1
24	1
26	1
27	3
29	1
33	1
35	1
41	1
47	1
50	2
74	1
76	1
82	1

CASE 42

VAL.	FRQ.
0	16
1	4
2	9
3	2
4	1
5	3
6	2
7	1
8	3
10	2
12	3
16	2
20	1
35	1

CASE 43

VAL.	FRQ.
17	1
21	1
33	1
40	1
43	1
45	1
50	1
51	2
52	1
55	1
57	1
61	1
63	2
64	1
68	1
69	1
71	1
76	1
78	1
80	1
82	1
83	1
87	1
90	1
93	1
94	1
95	1
98	1
100	1
101	1
102	1
104	1
109	1
114	2
121	1
125	1
127	1
129	1
135	1
140	1
149	1
158	1
163	1
167	1
171	1
184	1
185	1

CASE 44

VAL.	FRQ.
7	1
8	2
12	1
13	1
15	1
21	3
22	1
24	2
25	2
27	1
29	3
30	2
32	1
35	2
38	1
40	1
41	1
42	1
43	1
44	1
45	1
46	1
47	1
48	1
49	1
51	2
53	1
55	1
56	1
70	1
73	1
77	1
82	1
85	1
86	1
105	1
111	1
130	1
140	2

CASE 45

VAL.	FRQ.
0	12
1	4
2	3
3	5
4	2
6	1
7	1

CASE 45
(Cont'd)

VAL.	FRQ.
8	3
9	1
10	4
11	3
12	1
13	1
14	1
23	2
26	1
31	1
34	1
38	1
59	1
64	1

CASE 46

VAL.	FRQ.
0	35
1	32
2	20
3	7
4	5
5	1

CASE 47

VAL.	FRQ.
0	59
1	27
2	11
3	2
4	1

CASE 48

VAL.	FRQ.
0	92
1	7
2	1

CASE 49

VAL.	FRQ.
0	1
1	8
2	12
3	13
4	13
5	17
6	11
7	9
8	4
9	2
10	2
11	5
15	2
20	1

CASE 50

VAL.	FRQ.
0	15
1	21
2	22
3	17
4	12
5	5
6	2
7	2
8	3
13	1

CASE 51

VAL.	FRQ.
0	70
1	18
2	8
3	1
4	2
6	1

CASE 52

VAL.	FRQ.
0	1
2	3
3	3
4	4
5	4
6	3
7	7
8	12
9	7

CASE 52 (Cont'd)

VAL.	FRQ.
10	7
11	2
12	5
13	5
14	4
15	5
16	5
17	4
18	1
19	1
20	9
21	1
22	2
24	2
26	1
27	1
29	1

CASE 53

VAL.	FRQ.
0	5
1	9
2	3
3	10
4	16
5	9
6	7
7	7
8	7
9	3
10	7
11	5
12	2
13	2
14	2
15	2
17	1
19	1
23	1
24	1

CASE 54

VAL.	FRQ.
0	60
1	15
2	10
3	4
4	6
5	2
6	1
8	1
14	1

CASE 55

VAL.	FRQ.
13	1
15	3
18	1
19	1
22	1
23	1
24	2
25	4
26	2
28	1
29	1
30	1
31	1
32	2
33	2
34	1
35	3
36	6
37	1
38	4
39	3
40	2
41	1
42	2
43	1
44	2
45	1
46	3
47	2
49	1
50	2
51	1
52	1
53	3
54	2
55	2

CASE 55 (Cont'd)

VAL.	FRQ.
56	1
59	5
61	1
62	2
64	1
65	1
68	3
72	4
74	1
76	1
77	1
78	1
81	2
88	1
90	1
91	1
97	2
101	1
112	1
118	1

CASE 56

VAL.	FRQ.
1	1
4	2
5	7
6	1
7	2
8	3
9	2
10	4
11	2
12	3
13	2
14	4
15	2
16	4
17	3
18	2
19	2
20	4
21	1
22	3
23	4
24	5
25	2
26	1

CASE 56 (Cont'd)

VAL.	FRQ.
27	4
28	2
29	2
30	2
31	3
33	3
34	3
35	1
36	1
37	1
38	1
39	2
40	2
46	1
48	1
50	1
53	1
54	1
72	1
76	1

CASE 57

VAL.	FRQ.
0	32
1	10
2	10
3	6
4	9
5	7
6	2
7	2
8	4
9	1
10	2
12	2
13	1
14	1
17	3
18	3
21	1
23	1
27	1
29	1
35	1

CASE 58

VAL.	FRQ.
21	1
33	2
35	1
37	1
43	1
44	2
45	4
46	4
50	1
53	1
54	1
55	2
57	1
60	1
64	1
65	1
68	1
69	3
71	1
72	2
73	1
74	2
76	2
77	2
78	1
79	1
80	1
81	1
82	1
83	1
85	2
88	2
90	2
91	1
93	1
95	1
96	2
97	1
99	1
100	3
101	3
103	1
104	1
105	1
108	2
109	2
110	1
114	1

CASE 58 (Cont'd)

VAL.	FRQ.
116	1
118	2
119	1
121	1
122	1
124	1
125	1
127	2
131	1
135	1
137	1
140	1
141	1
144	1
147	1
152	1
153	1
161	1
180	1
185	2
190	1
192	1
193	1
197	1

CASE 59

VAL.	FRQ.
7	2
10	1
11	2
12	1
13	1
15	1
17	1
18	1
19	1
20	2
21	2
22	4
23	3
24	2
25	2
26	1
27	2
28	1
29	3

CASE 59 (Cont'd)

VAL.	FRQ.
30	3
32	3
34	2
35	5
36	1
37	2
38	1
39	1
40	2
41	2
42	2
44	1
45	1
47	1
49	2
51	1
52	1
53	1
54	1
56	1
57	2
58	1
61	1
63	1
64	1
65	1
66	1
67	2
69	1
70	1
72	1
73	1
78	2
81	1
82	3
85	2
88	1
90	1
91	1
93	2
98	1
102	1
103	1
107	1
110	1
140	1

VAL.	FRQ.
	CASE 60
0	18
1	10
2	7
3	6
4	5
5	4
6	1
7	3
8	2
9	4
10	1
11	5
12	1
13	1
14	2
15	1
16	4
17	1
18	2
19	2
20	2
21	2
22	2
25	1
26	3
28	1
30	2
31	2
32	1
38	1
50	1
53	1
90	1

APPENDIX C

COMPUTER PROGRAMS

Computer programs, to prepare Table (4.1) and Table (4.2) (Computer program 1), to find the estimates for the examples one through three of chatper three (computer program 2), and to generate the data for the sixty cases of the example four of chapter three and to find the estimates for them (computer program 3), are given in the following pages. There are two subroutines, namely GOLDEN (Golden section search), SVS (Single variable search), and one function, namely FCN (to evaluate efficiency function), which are common to the three programs mentioned above. These are given at the beginning under the title COMMON SUBROUTINES. After that the three programs are given.

One who wants to use any one of the programs, must include the common subroutines after the main program of the computer program he chooses.

COMMON SUBROUTINES

```
      SUBROUTINE SVS(FCN,TNOT)
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON P,R,DETLAM
      Q=1.0DO-P
      UPPER=1.0DO/DABS(Q)
      ALOW=-1.ODO/DSQRT(DABS(Q))
      RIOU=0.000001D0
      START=0.9999
      U=START
      S=0.0001D0
      A=FCN(U)
      U1=U+S
      B=FCN(U1)
      IF (B.GT.A) GO TO 5
      IF (B.EQ.A) GO TO 6
      M=3
1     U2=START+S*2.0D0**(M-2)
      IF (U2.GE.UPPER) GO TO 99
      A=B
      B=FCN(U2)
      IF(A.EQ.B) GO TO 7
      IF (A.LT.B) GO TO 8
      M=M+1
      U=U1
      U1=U2
         GO TO 1
7     CALL GOLDEN(FCN,U1,U2,RIOU,50,TNOT)
      GO TO 999
99    CALL GOLDEN(FCN,U1,UPPER,RIOU,50,TNOT)
      GO TO 999
8     CALL GOLDEN(FCN,U,U2,RIOU,50,TNOT)
      GO TO 999
6     M=2
      CALL GOLDEN(FCN,U,U1,RIOU,50,TNOT)
      GO TO 999
5     M=3
2     U2=START-S*2.0D0**(M-3)
      IF (U2.LE.ALOW) GO TO 77
      B=A
      A=FCN(U2)
      IF (A.EQ.B)  GO TO 17
      IF (A.GT.B)  GO TO 18
      U1=U
      U=U2
      M=M+1
      GO TO 2
17    CALL GOLDEN(FCN,U2,U,RIOU,50,TNOT)
      GO TO 999
18    CALL GOLDEN(FCN,U2,U1,RIOU,50,TNOT)
      GO TO 999
77    CALL GOLDEN(FCN,ALOW,U,RIOU,50,TNOT)
999   RETURN
      END
```

```
      SUBROUTINE GOLDEN(FCN,ENDL,ENDR,RIOU,NMAX,XOLD)
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON P,R,DETLAM
      SIGMA=(3.0D0-DSQRT(5.ODO))/2.0D0
      AL=ENDR-ENDL
      XOLD=ENDL+SIGMA*AL
      FOLD=FCN(XOLD)
      DO 1 I=2,NMAX
      XNEW=ENDL+ENDR-XOLD
      FNEW=FCN(XNEW)
      IF (FNEW.LE.FOLD) GO TO 5
      IF (XNEW.LT.XOLD) GO TO 6
      ENDR=XNEW
20    AL=ENDR-ENDL
      GO TO 30
6     ENDL=XNEW
      GO TO 20
5     IF (XNEW.LT.XOLD) GO TO 7
      ENDL=XOLD
8     XOLD=XNEW
      FOLD=FNEW
      GO TO 20
7     ENDR=XOLD
      GO TO 8
30    IF(AL.LE.RIOU) GO TO 40
1     CONTINUE
40    RETURN
      END

      FUNCTION FCN(X)
      IMPLICIT REAL *8 (A-H,O-Z)
      COMMON P,R,DETLAM
      ZR = 1.0D-73
      Q=1.0D0-P
      A=R*P**(R+2.DO)/Q
      B=1.0D0-Q*X
      C=(DABS((B*DLOG(B/P)/P-Q*(1.0D0-X)/P)))**2.D0
      D=B**(2.D0*R+2.0D0)/(1.0D0-Q*DABS(X)**2.D0)**R
      E=P**R*B**2.D0+R*Q*P**R*(DABS(1.0D0-X))**2.D0
      G=-A*C/DETLAM
      F=D-E
      IF (DABS(F).LT.ZR) GO TO 5
      FCN=G/F
      GO TO 7
5     FCN=-1.0D0
7     RETURN
      END
```

COMPUTER PROGRAM 1

```
      IMPLICIT REAL*8 (A-H,O-Z)
      EXTERNAL FCN
      DIMENSION PR(14),RR(26),EFF(26,14),TZERO(26,14),TTEST(52),EFF1(26,
     114),EFF2(26,14)
      COMMON P,R,DETLAM
      DATA EFF,TZERO/728*0.0D0/
      DATA EFF1,EFF2/728*0.0D0/
      DATA PR/.05D0,.06D0,.07D0,.08D0,.09D0,.1D0,.2D0,.3D0,.4D0,.5D0,.6D
     10,.7D0,.8D0,.9D0/
      DATA RR/.5D0,1.D0,1.5D0,2.D0,2.5D0,3.D0,3.5D0,4.D0,4.5D0,5.D0,6.D0
     1,7.D0,8.D0,9.D0,10.D0,11.D0,12.D0,13.D0,14.D0,15.D0,16.D0,17.D0,18
     1.D0,19.D0,0.0D0,0.0D0/
      DATA TTEST/52*0.0D0/
      WRITE (6,444)
444   FORMAT('1')
      EPS=0.0001D0
      DO 1 J=1,14
      P=PR(J)
      Q=1.0D0-P
      K=1
7     TEST=Q**(K+1)/(P*(DFLOAT(K)+2.0D0))
      K=K+1
      GO TO 7
8     M=K
      WRITE (6,21) M
21    FORMAT(15X,'UPPER BOUND=',I8)
      DO 2 I=1,24
      R=RR(I)
      DETLAM=0.0D0
      DO 10 N=1,M
      RATION=1.0D0
      DO 12 JJ=1,N
12    RATION=RATION*DFLOAT(JJ)/(DFLOAT(JJ)+R)
      DETLAM=DETLAM+Q**N*RATION/DFLOAT(N+1)
10    CONTINUE
      CALL SVS(FCN,TNOT)
      TZERO(I,J)=TNOT
      TPL=DEXP(0.5D0*DLOG(TNOT))
      TMIN=DEXP(1.5D0*DLOG(TNOT))
      EFF2(I,J)=-FCN(TPL)
      EFF1(I,J)=-FCN(TMIN)
      EFF(I,J)=-FCN(TNOT)
2     CONTINUE
1     CONTINUE
      WRITE (6,200)  (PR(I),I=1,14)
200   FORMAT(///,50X,'P  VALUES',/,12X,14(2X,F5.3,1X),/,12X,14(2X,'-----
     1',1X),/)
      DO 13 I=1,24
      WRITE (6,202) (EFF1(I,J),J=1,14)
      WRITE (6,201) RR(I),(EFF(I,J),J=1,14)
      WRITE (6,202) (EFF2(I,J),J=1,14)
```

```
201     FORMAT(3X,'R=',F5.2,2X,14(2X,F5.3,1X))
        WRITE (6,203) (TZERO(I,J),J=1,14)
202     FORMAT(12X,14(1X,'(',F5.3,')'))
203     FORMAT(5X,'OP.TNOT',14(2X,F5.3,1X),//)
13      CONTINUE
        STOP
        END
```

COMPUTER PROGRAM 2

```
      IMPLICIT REAL*8(A-H,O-Z)
      EXTERNAL FCN
      DIMENSION FRQ(20),OBS(20)
      COMMON P,R,DETLAM
      WRITE (6,444)
444   FORMAT('1')
      EPS=0.0001D0
      WRITE (6,91)
91    FORMAT(5X,'READ IN NO. OF OBS. AND VARIATES IN 2I5 FORMAT',/)
      READ (5,81) NOBS,.NVAR
81    FORMAT(2I5)
      WRITE (6,92)
92    FORMAT(5X,'NOW GIVE ME VARIATE VALUES AND CORRESPONDING FREQUENCY
     1IN THE FORMAT 2F10.5 FOR EACH CARD',/)
      SUM=0.0D0
      SUM1=0.0D0
      DO 4 I=1,NVAR
      READ (5,82) CBS(I),FRQ(I)
82    FORMAT(2F10.5)
      SUM=SUM+OBS(I)*FRQ(I)
      SUM1=SUM1+(OBS(I)**2.0D0)*FRQ(I)
      XBAR=SUM/DFLOAT(NOBS)
      SSQR=(SUM1-DFLOAT(NOBS)*XBAR**2.0D0)/DFLOAT(NOBS-1)
      WRITE (6,209) XBAR, SSQR
209   FORMAT(15X,'USING GIVEN DATA FOLLOWING INFORMATION IS OBTAINED',//
     1,20X,'SAMPLE MEAN       =',F7.4,//,20X,'SAMPLE VARIANCE  =',F7.4,//
     2)
      P=XBAR/SSQR
      R=XBAR**2.0D0/(SSQR-XBAR)
      Q=1.0D0-P
      K=1
7     TEST=Q**(K+1)/(DFLOAT(K+2))
      IF(TEST.LE.EPS) GO TO 8
      K=K+1
      GO TO 7
8     M=K
      DETLAM=0.0D0
      DO 10 N=1,M
      RATION=1.0D0
      DO 12 JJ=1,N
12    RATION=RATION*DFLOAT(JJ)/(DFLOAT(JJ)+R)
      DETLAM=DETLAM+Q**N*RATION/DFLOAT(N+1)
10    CONTINUE
      L=1
95    CALL SVS(FCN,TNOT)
      EFF1=-FCN(TNOT)
      RSTAR=R
      PSTAR=P
      SUM2=0.0D0
      DO 14 J=1,NVAR
14    SUM2=SUM2+FRQ(J)*TNOT**OBS(J)
      TXBAR=SUM2/DFLOAT(NOBS)
```

```
      CALL COMPR(TNOT,TXBAR,XBAR,RSTAR)
      P=R/(R+XBAR)
      Q=1.0D0-P
      K=1
77    TEST=Q**(K+1)/(P*DFLOAT(K+2))
      IF (TEST.LE.EPS) GO TO 88
      K=K+1
      GO TO 77
88    M=K
      DETLAM=0.0D0
      DO 51 N=1,M
      RATION=1.0D0
      DO 52 JJ=1,N
52    RATION=RATION*DFLOAT(JJ)/(DFLOAT(JJ)+R)
51    DETLAM=DETLAM+Q**N*RATION/DFLOAT(N+1)
      EFF2=-FCN(TNOT)
      IF(DABS(P-PSTAR).GT.0.005) GO TO 29
      IF(DABS(R-RSTAR).GT.0.005) GO TO 29
      IF(DABS(EFF1-EFF2).GT.0.04) GO TO 29
      WRITE (6,94) P,R,EFF2,TNOT,L
      FORMAT(20X,'ESTIMATE OF P    =',F7.5,//,20X,'ESTIMATE OF R    =',F
     17.4,//,20X,EFFICIENCY         =',F7.5,//,20X,'OPTIMUM TNOT     =',F
     27.5,//,20X,'NO.OF ITERATIONS =',FI4,//)
      GO TO 999
29    L=L+1
      GO TO 95
999   STOP
      END

      SUBROUTINE COMPR(T,TX,XB,RST)
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON P,R,DETLAM
      R1=RST
30    B+R1+XB*(1.0D0-T)
      TY=(R1/B)**R1
      F=TX-TY
       FP1=-TY
       FP2=(XB*(1.0D0-T)/B)+DLOG(R1/B)
       FP=FP1*FP2
       R2=R1-F/FP
      IF(DABS(R2-R1).LE.0.00001D0) GO TO 66
      R1=R2
      GO TO 30
66    R=R2
      RETURN
      END
```

COMPUTER PROGRAM 3

```
      IMPLICIT REAL*8 (A-H,O-Z)
      EXTERNAL FCN
      DIMENSION FRQ(200),OBS(200),PVECT(10),RVECT(10),NVECT(10),DAT(200)
     1,ISD(200)
      COMMON P,R,DETLAM
      DATA RVECT/5.0D0,2.5D0,0.5D0,7*0.0D0/
      DATA PVECT/.8D0,.5D0,.3D0,.1D0,.05D0,5*0.0D0/
      DATA NVECT/15,30,50,100,6*0/
      CALL OVFLOW
      WRITE(5,541)
541   FORMAT(//,15X,'READ ISEED USED IN GENERATING DATA')
      READ(5,978)  IS
978   FORMAT(I10)
      WRITE(6,979)  IS
979   FORMAT(15X,'ISEED USED TO GENERATE DATA=',I10)
      CALL INT(IS,ISD,200)
      WRITE (6,444)
444   FORMAT('1')
      EPS=0.0001D0
      NCASE=0
      DO 112 IN=1,4
      DO 113 IP=1,5
      DO 114 IR=1,3
      P=PVECT(IP)
      R=RVECT(IR)
      NOBS=NVECT(IN)
      NCASE=NCASE+1
      NCP=2*NCASE
      NCM=NCP-1
      ISEED1=ISD(NCP)
      ISEED2=ISD(NCM)
      CALL GNRNB(NOBS,OBS,P,R,ISEED2,ISDEED1)
      OBMX=-99999.D0
      DO 321 LK=1,NOBS
      IF (OBS(LK).GT.OBMX)  OBMX=OBS(LK)
321   CONTINUE
      OBLM=OBMX+10.D0
      NVAR=0
390   OBMN=OBMX+5.D0
      DO 370 MI=1,NOBS
      IF (OBS(MI).LT.OBMN)  OBMN=OBS(MI)
370   CONTINUE
      ICNT=0
      DO 380 MJ=1,NOBS
      IF (OBS(MJ).NE.OBMN)  GO TO 380
      ICNT=ICNT+1
      OBS(MJ)=OBLM
380   CONTINUE
      NVAR=NVAR+1
      DAT(NVAR)=OBMN
      FRQ(NVAR)=DFLOAT(ICNT)
      IF(OBMN.EQ.OBMX)  GO TO 395
```

```
      GO TO 390
395   WRITE (6,889)
889   FORMAT(//,10X,89('*'),/,40X,'DATA GENERATED IS GIVEN BELOW',/,39X,
     131('*'),/)
      WRITE (6,301)  (DAT(LO),LO=1,NVAR)
301   FORMAT(20X,'VALUES',/,20X,6('-'),/,(5X,20(1X,F5.0),/))
      WRITE (6,302)  (FRQ(LO),LO=1,NVAR)
302     format920X,'FREQUENCIES',/,20X,11('-'),/,(5X,20(1X,F5.0),/))
      WRITE(6,543)
543   FORMAT(//,10X,'DATA WRITTEN ABOVE IS BASED ON FOLLOWING PARAMETERS
     1',/,10X,52('-'))
      WRITE (6,544) NCASE,P,R,NOBS,NVAR
544   FORMAT(/,10X,'CASE NO.=',I5.5X,'P=',F7.4,5X,'R=',F7.2,5X,'N=',Ir,5
     1X,'NO.OF VALUES=',I3,/)
      SUM=0.0D0
      SUM1=0.0D0
      DO 445 I=1,NVAR
      SUM=SUM+FRQ(I)*DAT(I)
445   SUM1=SUM1+FRQ(I)*DAT(I)**2.0D0
      XBAR=SUM/DFLOAT(NOBS)
      SSQR=(SUM1-DFLOAT(NOBS)*XBAR**2.0D0)/DFLOAT(NOBS-1)
      IF (XBAR.GE.SSQR) GO TO 71
      P=XBAR/SSQR
      R=XBAR**2.0D0/(SSQR-XBAR)
      GO TO 72
71    R=5.0D0
      P=0.9D0
      GO TO 114
72    Q=1.0D0-P
      K=1
7     TEST=Q**(K+1)/(DFLOAT(K+2))
      IF(TEST.LE.EPS) GO TO 8
      K=K+1
      GO TO 7
8     M=K
      DETLAM+0.0D0
      DO 10 N=1,M
      RATION=1.0D0
      DO 12 JJ=1,N
12    RATION=RATION*DFLOAT(JJ)/(DFLOAT(JJ)+R)
      DETLAM=DETLAM+Q**N*RATION/DFLOAT(N+1)
10    CONTINUE
      L=1
95    CALL SVS(FCN,TNOT)
      EFF1=-FCN(TNOT)
      RSTAR=R
      PSTAR=P
      SUM2=0.0D0
      DO 145 J=1,NVAR
145   SUM2=SUM2+FRQ(J)(TNOT**DAT(J)
      TXBAR=SUM2/DFLOAT(NOBS)
      CALL COMPR(TNOT,TXBAR,XBAR,RSTAR)
```

```
      P=R/(R+XBAR)
      Q=1.0D0-P
      K=1
77    TEST=Q**(K+1)/(P*DFLOAT(K+2))
      IF (TEST.LE.EPS) GO TO 88
      K=K+1
      GO TO 77
88    M=K
      DETLAM=0.0D0
      DO 51 N=1,M
      RATION=1.0D0
      DO 52 JJ=1,N
52    RATION=RATION*DFLOAT(JJ)/(DFLOAT(JJ)+R)
51    DETLAM=DETLAM+Q**N*RATION/DFLOAT(N+1)
      EFF2=-FCN(TNOT)
      IF (L.NE.1) GO TO 141
      UNOT=-DLOG(DABS(TNOT))
      WRITE(6,98) PSTAR,RSTAR,P,R,EFF1,EFF2,UNOT
98    FORMAT(/,11X,57('&'),/,15X,'VALUES OBTAINED AT THE END OF FIRST IT
     1ERATION',/,12X,'PTHIL',3X,'RTHIL',3X,'PSTAR',3X,'RSTAR',3X,'ETHIL'
     2,3X,'ESTAR',3X,'UNOT',/,12X,F5.3,2X,F7.3,2X,F5.3,2X,F7.3,2X,F5.3,3
     3X,F5.3,2X,F7.4,/,11X,57('&'),/)
141   IF(DABS(P-PSTAR).GT.0.001) GO TO 29
      IF(DABS(R-RSTAR).GT.0.005) GO TO 29
      IF(DABS(EFF1-EFF2).GT.0.01) GO TO 29
      UNOT=-DLOG(DABS(TNOT))
      WRITE (6,94) XBAR,SSQR,P,R,EFF2,UNOT,L
94    FORMAT(15X,'USING THE ABOVE DATA FOLLOWING INFORMATION IS OBTAINED
     1/,/,15X,54('-'),/,10X,'SPLE.MEAN',2X,'SPLE.VAR',3X,'EST.OF P',3X,'
     2EST.OF R',6X,'EFF',6X,'OP.UNOT',3X,'NO.OF ITR',/,10X,7(9('-'),2X),
     3/,10X,F9.3,2X,F9.3,2X,F9.3,2X,F9.3,2X,F9.3,2X,F9.4,5X,I3,//)
      GO TO 114
29    L=L+1
      GO TO 95
114   CONTINUE
113   CONTINUE
112   CONTINUE
      STOP
      END
```

```
      SUBROUTINE GNRNB(N,OBS,P,R,ISEED2,ISEED1)
      REAL*8 P,R,OBS
      DIMENSION OBS(200),GAM(200),K(1)
      P1=SNGL(P)
      R1=SNGL(R)
      CALL GAMA(R1,ISEED1,GAM,N)
      DO 1 I=1,N
      RLAM=GAM(I)*(1.-P1)/P1
      CALL GGPOSH(RLAM,ISEED2,1,K,IER)
1     OBS(I)=DBLE(FLOAT(K(1)))
      RETURN
      END
```

```
      SUBROUTINE COMPR(T,TX,XB,RST)
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON P,R,DETLAM
      R1=RST
30    B=R1+XB*(1.0D0-T)
      TY=DABS(R1/B)**R1
      F=TY-TX
       FP1=TY
       FP2=(XB*(1.0D0-T)/B)+DLOG(DABS(R1/B))
       FP=FP1*FP2
       R2=R1-F/FP
      IF(DABS(R2-R1).LE.0.0001D0) GO TO 66
      R1=R2
      GO TO 30
66    R=R2
      RETURN
      END
```

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